A remark on least-squares mixed element methods for reaction–diffusion problems

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Abstract

In this paper, we propose a least-squares mixed element procedure for a reaction–diffusion problem based on the first-order system. By selecting the least-squares functional properly, the resulting procedure can be split into two independent symmetric positive definite schemes, one of which is for the unknown variable and the other of which is for the unknown flux variable, which lead to the optimal order $H^1$ and $L^2$ norm error estimates for the primal unknown and optimal $H(\text{div}; \Omega)$ norm error estimate for the unknown flux. Finally, we give some numerical examples.

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1. Introduction

The purpose of this paper is to consider the least-squares mixed element approaches for a symmetric positive definite elliptic problem with reaction term written as a first-order system. It is well known that the least-squares mixed element method has two typical advantages as follows: it is not subjected to the Ladyzhenkaya–Babuska–Brezzi consistency condition (see [9,1,3]), so the choice of approximation spaces becomes flexible, and it results in a symmetric positive definite system.

An elegant theory of the least-squares mixed element methods for approximating elliptic boundary value problem, based on the first-order system, was introduced by Pehlivanov et al. [10], where a least-squares residual minimization is introduced for the mixed system in unknown variable $u$ and unknown velocity-flux $\sigma$. They also established the optimal $H^1$-norm error estimates for the unknown variable $u$ and $H(\text{div}; \Omega)$-norm error estimate for the unknown velocity-flux $\sigma$. Then, Cai et al. [4,5] extended the results in [10] to a least-squares method for second-order partial differential equations including the convection and reaction terms. By introducing a velocity-flux variable and associated curl and

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In this paper, for a kind of elliptic boundary value problem, that is, the reaction–diffusion problem, we introduce a least-squares mixed element scheme. By selecting the least-squares functional property, the resulting least-squares procedure can be split into two independent symmetric positive definite sub-schemes. The first sub-scheme is for the unknown variable $u$, which is the same as the standard Galerkin finite element approximation. The second sub-scheme is for the unknown flux $\sigma$. In this case we can select the approximation spaces for unknown variable $u$ and unknown flux $\sigma$ independently. For each of these sub-schemes, we give the optimal order error estimates. We also give some numerical examples using the split least-squares scheme. The numerical simulations are consistent with the theoretical results. In a forthcoming paper we will consider using this idea to deal with parabolic problems.

The remainder of this paper is organized as follows. In Section 2 for a kind of reaction–diffusion problem we introduce the split least-squares procedure and show continuity, coercivity and perform the error analysis. In Section 3 we give some numerical examples.

Throughout this paper, the notations of standard Sobolev spaces $L^2(\Omega)$, $H^k(\Omega)$ and associated norms $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$ are adopted as those in [7]. A constant $C$ (with or without subscript) stands for a generic positive constant independent of the mesh parameter $h$, which may appear differently at different occurrences.

2. A split least-squares mixed element procedure

Consider the following reaction–diffusion problem with a positive reaction term on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

\[
\begin{aligned}
-\text{div}(A\nabla u) + qu &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \Gamma,
\end{aligned}
\]

(2.1)

where $\Gamma = \partial \Omega$, $A = A(x) = (a_{ij}(x))_{i,j=1}^d$ is a bounded, symmetric and positive definite matrix in $\Omega$, i.e., there exist positive constants $\alpha_1$ and $\alpha_2$ such that,

\[
\alpha_1 \|\xi\|^2 \leq (A \xi, \xi) \leq \alpha_2 \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^d.
\]

(2.2)

We further suppose that $q$ is positive definite and is bounded, that is, there exist a positive constant $C_q$ and a positive constant $q_0$ such that

\[
C_q \geq q \geq q_0 > 0 \quad \text{on } \Omega.
\]

(2.3)

Under this assumption, we can describe a least-squares mixed element method which can be divided into two independent problems.

Define two Hilbert spaces as

\[
V = H^1_0(\Omega),
\]

(2.4)

\[
W = \{ \tau \in (L^2(\Omega))^d : \text{div } \tau \in L^2(\Omega) \}.
\]

(2.5)

Introducing the flux $\sigma = -A\nabla u$, $\sigma = (\sigma_1, \ldots, \sigma_d)$, the problem (2.1) can be rewritten as a first-order system: find $(u, \sigma) \in V \times W$ such that

\[
\begin{aligned}
\text{div } \sigma + qu - f &= 0 & \text{in } \Omega, \\
\sigma + A\nabla u &= 0 & \text{in } \Omega,
\end{aligned}
\]

(2.6)

with the same boundary conditions as in (2.1). System (2.6) appears in many realistic applications, such as the flow problem in porous media, where $u$ denotes the pressure and $\sigma$ denotes the Darcy velocity. At these cases the approximations to both $u$ and $\sigma$ are necessary.
By multiplying the first equation by \( q^{-1/2} \) and the second equation by \( \tilde{A} = A^{-1} \) in (2.6), we have the equivalent first-order system of equations satisfying
\[
\begin{align*}
q^{-1/2}(\text{div } \sigma + qu - f) &= 0 \quad \text{in } \Omega, \\
\tilde{A}^{1/2}(\sigma + A\nabla u) &= 0 \quad \text{in } \Omega,
\end{align*}
\]
with the same boundary conditions as in (2.1). For \((v, \tau) \in V \times W\), define the least-squares functional \(J(v, \tau)\) as follows:
\[
J(v, \tau) = \|q^{-1/2}(\text{div } \tau + qv - f)\|^2 + \|\tilde{A}^{1/2}(\tau + A\nabla v)\|^2.
\]
(2.8)
Then the least-squares minimization problem corresponding to (2.7) is to find a solution \((u, \sigma) \in V \times W\) such that
\[
J(u, \sigma) = \inf_{v \in V, \tau \in W} J(v, \tau).
\]
(2.9)
Define the bilinear form \(a(\cdot, \cdot; \cdot, \cdot)\) corresponding to the least-squares functional \(J\) as
\[
a(u, \sigma; v, \tau) = (q^{-1}(\text{div } \sigma + qu), \text{div } \tau + qv) + (\tilde{A}(\sigma + A\nabla u), \tau + A\nabla v)
\]
\[
= (q^{-1}(\text{div } \sigma + u), \text{div } \tau + qv) + (\tilde{A} \sigma + \nabla u, \tau + A\nabla v).
\]
(2.10)
Now the weak statement of the minimization problem (2.9) becomes: find \((u, \sigma) \in V \times W\) such that
\[
a(u, \sigma; v, \tau) = (q^{-1} f, \text{div } \tau + qv) \quad \forall (v, \tau) \in V \times W.
\]
(2.11)
For a finite element approximation, let \(\mathcal{T}_{h_u}\) and \(\mathcal{T}_{h_\sigma}\) be two families of regular finite element partitions of the domain \(\Omega\), which could be identical or not. Here, \(h_u\) and \(h_\sigma\) denote the largest diameters of elements in \(\mathcal{T}_{h_u}\) and \(\mathcal{T}_{h_\sigma}\), respectively. Based on \(\mathcal{T}_{h_u}\) and \(\mathcal{T}_{h_\sigma}\), we construct the finite element spaces \(V_h \subset V\) and \(W_h \subset W\) with the following approximation properties:
\[
\inf_{v_h \in V_h} \|v - v_h\| + h_u\|\nabla(v - v_h)\| \leq C h_{m+1}^m \|v\|_{m+1},
\]
(2.12)
\[
\inf_{\tau_h \in W_h} \|\tau - \tau_h\| \leq C h_{k+1}^k \|\tau\|_{k+1},
\]
(2.13)
\[
\inf_{\tau_h \in W_h} \|\text{div}(\tau - \tau_h)\| \leq C h_{k+1}^k \|\tau\|_{k+1},
\]
(2.14)
for \(v \in V \cap H^{m+1}(\Omega), \tau \in W \cap (H^{k+1}(\Omega))^d\). It is clear that in (2.14) we have \(k_1 = k\) at least if (2.13) holds, and \(k_1 = k + 1\) if \(W_h\) is selected as the Raviart–Thomas mixed element space of index \(k\) (see [11]).

Based on (2.11), the least-squares mixed finite element approach reads as follows.

**Scheme (I).** Find \(u_h \in V_h, \sigma_h \in W_h\) such that
\[
a(u_h, \sigma_h; v_h, \tau_h) = (q^{-1} f, \text{div } \tau_h + qv_h) \quad \forall (v_h, \tau_h) \in V_h \times W_h.
\]
(2.15)
Now we will discuss the bilinear form \(a(u, \sigma; v, \tau)\) in the following lemma, which leads to a decoupled system.

**Lemma 2.1.** For \(u, v \in V\) and \(\sigma, \tau \in W\) we have that
\[
a(u, \sigma; v, \tau) = (q^{-1} \text{div } \sigma, \text{div } \tau) + (qu, v) + (\tilde{A} \sigma, \tau) + (A \nabla u, \nabla v).
\]
(2.16)
**Proof.** A direct calculation shows that for \(\sigma, \tau \in W\) and \(u, v \in V\)
\[
a(u, \sigma; v, \tau) = (q^{-1} \text{div } \sigma, \text{div } \tau) + (qu, v) + (\tilde{A} \sigma, \tau) + (A \nabla u, \nabla v)
\]
\[
+ (\text{div } \sigma, v) + (u, \text{div } \tau) + (\nabla u, \tau) + (\sigma, \nabla v),
\]
(2.17)
Theorem 2.4. Assume that problem (2.1) is \( H^2 \)-regular. Let \((u, \sigma) \in H^{m+1} \times H^{k_1+1} \) be the exact solution to (2.6). Let \((u_h, \sigma_h)\) be the solution of (2.15), then there exists a positive constant \( C \) independent of \( h_u \) and \( h_\sigma \) such that
\[
\| u_h - u \| \leq C h_u^{m+1-s} \| u \|_{m+1}, \quad s = 0, 1,
\]
and
\[
\| \sigma_h - \sigma \|_{H^{k_1+1}} \leq C h_\sigma^{k_1} \| \sigma \|_{k_1+1}.
\]

Proof. From the error estimates of Galerkin finite element methods for elliptic problems (see [3]), it is obviously that (2.22) holds for \( s = 1 \). Since problem (2.1) is \( H^2 \)-regular, (2.22) holds for \( s = 0 \).

Next, we prove (2.23). With \( v = 0 \) in (2.11), from Lemma 2.1 we have
\[
(q^{-1} \text{div } \sigma, \text{div } \tau_h) + (\tilde{A} \sigma_h, \tau_h) = (q^{-1} f, \text{div } \tau_h) \quad \forall \tau_h \in W_h.
\]
Subtracting (2.24) from (2.20) we have that for all \( \tau_h \in W_h \),
\[
(q^{-1} \text{div}(\sigma_h - \sigma_I), \text{div } \tau_h) + (\tilde{A}(\sigma_h - \sigma_I), \tau_h) = (q^{-1} \text{div}(\sigma - \sigma_I), \text{div } \tau_h) + (\tilde{A}(\sigma - \sigma_I), \tau_h),
\]
where $\sigma_I \in W_h$ is an interpolant of $\sigma$ satisfying
\[
\|\sigma - \sigma_I\|_{H(\text{div}; \Omega)} \leq C h_{\sigma}^{k_1} \|\sigma\|_{k+1}.
\] (2.25)

Let $\tau_h = \sigma_h - \sigma_I$. Using Lemma 2.3, we have
\[
\|\sigma_h - \sigma_I\|_{H(\text{div}, \Omega)}^2 \leq C \|\sigma_h - \sigma_I\|_{H(\text{div}; \Omega)} \|\sigma - \sigma_I\|_{H(\text{div}; \Omega)},
\]
\[
\|\sigma_h - \sigma_I\|_{H(\text{div}; \Omega)} \leq C h_{\sigma}^{k_1} \|\sigma\|_{k+1}.
\] (2.26)

Combining (2.26) with (2.25) completes the proof. □

Remark 2.5. When a homogeneous boundary condition is replaced by a nonhomogeneous boundary condition as follows,
\[u = g_1 \text{ on } \Gamma,
\]
it follows that for $v \in H^1_0(\Omega), \tau \in H(\text{div}; \Omega)$
\[(\text{div } \sigma, v) + (u, \text{div } \tau) + (\nabla u, \tau) + (\sigma, \nabla v) = \int_{\partial \Omega} \sigma \cdot n v \, ds + \int_{\partial \Omega} u \tau \cdot n \, ds
\]
\[= \int_{\partial \Omega} g_1 \tau \cdot n \, ds
\]
then the least-square method can also be split into two independent sub-procedures. In this case we have to subtract the above boundary integral from the right-hand side of (2.20).

Remark 2.6. The same idea can be used to deal with the initial-boundary value problem for the time-dependent parabolic equation
\[u_t - \nabla \cdot (A \nabla u) = f.
\]
After time discretization it becomes
\[
\frac{1}{\Delta t} u^n - \nabla \cdot (A \nabla u^n) = F,
\]
where $F = (1/\Delta t) u^{n-1} + f^n$. This case corresponds to the reaction–diffusion problem with $q = \Delta t^{-1}$. A detailed discussion will be considered in a forthcoming paper.

3. Numerical examples

In this section we give some numerical examples using Scheme (I) constructed in Section 2. Since the sub-procedure for $u$ is the standard Galerkin finite element procedure, it is sufficient to give some numerical examples using the sub-procedure for the flux $\sigma = A \nabla u = (\sigma_1, \sigma_2)$ to get the approximation $\sigma_h = (\sigma_{h,1}, \sigma_{h,2})$. We consider the following problem:
\[\text{div}(A \nabla u) + qu = f \text{ in } \Omega.
\]
For simplicity $\Omega$ is selected as a square in two-dimensional domain, $A$, $q$ are constant matrix and constant, respectively. Divide $\Omega$ into $N \times N$ squares with mesh size $h$. Based on this triangulation we select $W_h$ as the lowest order Raviart–Thomas mixed element space [11]. The set of vertices for $\sigma_1$ is $\{x_{i,j+1/2} = (ih, (j + 1/2)h)\}$, and the set of vertices for $\sigma_2$ is $\{x_{i+1/2,j} = ((i + 1/2)h, jh)\}$.
Now we give two numerical examples. In the first example, the analytical solution is \( u = \sin(x) \sin(y) \), and \( \Omega = (0, \pi) \times (0, \pi) \). For a set of simulations, different mesh sizes and different values of \( q \) are taken and their corresponding errors are listed in Table 1. Here the \( e_{\sigma, l^{\infty}} \) and \( e_{\sigma, l^2} \) are defined as

\[
e_{\sigma, l^{\infty}} = \max_{i,j} \{ |\sigma_1(x_{i,j+1/2}) - \sigma_{h,1}(x_{i,j+1/2})|, |\sigma_2(x_{i+1/2,j}) - \sigma_{h,2}(x_{i+1/2,j})| \},
\]

\[
e_{\sigma, l^2} = h \left( \sum_{i,j} |\sigma_1(x_{i,j+1/2}) - \sigma_{h,1}(x_{i,j+1/2})|^2 + \sum_{i,j} |\sigma_2(x_{i+1/2,j}) - \sigma_{h,2}(x_{i+1/2,j})|^2 \right)^{1/2}.
\]

In the second example, the analytical solution is \( u = x(1-x)y(1-y) \exp(x - y) \), and \( \Omega = (0, 1) \times (0, 1) \). For different mesh sizes and different values of \( q \), the errors are listed in Table 2.

These numerical examples show the convergence of flux \( \sigma \). Even for the smallest \( q \)-value, the numerical results are very good. Here, the computation of \( \sigma_h \) is independent of the approximation of \( u \).

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References


